1. Define μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by letting $\mu(A)$ be the number of rational numbers in A (of course $\mu(A) = +\infty$ if there are infinitely many rational numbers in A). Show that μ is a σ-finite measure. (5p)

2. On a measure space with measure μ we have measurable sets $\{A_n\}_{n\geq 1}$, A, and B. Recall that a sequence of measurable functions $\{f_n\}$ converges to f in μ-measure, if for all $\varepsilon > 0$, $\mu(|f_n - f| > \varepsilon) \to 0$, $n \to \infty$. Assume first that the sequence of indicator functions $\{1_{A_n}\}$ converges μ-*almost everywhere* to 1_A *and* converges in μ-*measure* to 1_B. Prove that $\mu(B) = \mu(A)$. Secondly, assume that all we know is that $\{1_{A_n}\}$ converges in μ-*measure* to 1_B. Prove that $\mu(B) = \mu(\liminf_n A_n)$. (6)

4. Let μ be a σ-finite measure on the real line, which is singular with respect to Lebesgue measure, $\mu \perp m$. Let A be a set with Lebesgue measure $m(A) = 0$. Show that almost all translates of A are zero sets under μ, that is, $\mu(A + x) = 0$, m-a.e. x. (In spite of the fact that there must exist some set A with $m(A) = 0$ and $\mu(A) > 0$, since μ is singular!) (6)

5. [redacted] real-valued function on [redacted] $[0, 1]$. (5p)

6. Show that a function f on the real interval $[0, 1]$ satisfies $|f(x) - f(y)| \leq M|x - y|$ for some $M < \infty$ and all $x, y \in [0, 1]$ if and only if there is a bounded measurable function g such that $f(x) = f(0) + \int_0^x g(y)\, dy$. (6p)

7. Suppose f is an absolutely continuous real-valued function on $\mathbb{R}$ with derivative $f' \in L^p(\mathbb{R})$, $1 \leq p < \infty$.

   a) Prove that for each $x \in \mathbb{R}$, $h^{1-p}(f(x+h) - f(x))^p \to 0 \quad h \to 0$. (3p)

   b) Under the additional assumption that $f(x) = 0$, $x \notin [0, 1]$, and we also have $f(0) = 0$, prove that

$$\int_0^1 |f(x)|^p\, dx \leq \frac{1}{p} \int_0^1 |f'(x)|^p\, dx$$

(3p)