On Possibilities to Vary Speed and Curvature in Trajectories Represented by B-Splines

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The starting point for the following considerations is the representation

$$z(t) = \begin{pmatrix} x(t) \\ y(y) \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} c_i \\ d_i \end{pmatrix} B_i(t), \quad t \in [t_1, t_m]. \tag{1}$$

Our intention consists of deriving the B-spline representation of a trajectory whose velocity and/or curvature is perturbed by given functions. This leads to two different problems:

1. Perturbation of the velocity, only

   Velocity is a vector function. Therefore, we must allow for perturbations of all components of this vector. Let $v = z'(t)$. Then we search for a trajectory $\tilde{z}(t) = z(t) + \Delta z(t)$ such that $\tilde{z}'(t) = v(t) + \Delta v(t)$ for a given Δv.

2. Perturbation of velocity and curvature

   This problem is slightly more intricate. Since the absolute value of velocity $|v|$ and the curvature κ determine the curve uniquely (up to a translation), it is these two values which should be changed to $|v| + \Delta v$ and $\kappa + \Delta\kappa$, respectively.

These two questions will lead to very different approaches.

Let us introduce some notation: By B_i, $i = 1, \dots, n$ we denote the B-spline basis of $S_{k,\mathbf{t}}$, that is $S_{k,\mathbf{t}} = \operatorname{lin}\{B_1, \dots, B_n\}$. Moreover, we introduce the linear space $S'_{k,\mathbf{t}} = \operatorname{lin}\{B'_1, \dots, B'_n\}$ containing all derivatives of splines in $S_{k,\mathbf{t}}$. Additionally, we denote by $P_{k\mathbf{t}} : C[t_1, t_m] \to S_{k,\mathbf{t}}$ a projector. At the moment, it can be any projector. Some choices will be discussed below.

1 Perturbation of the Velocity

The problem is defined as follows: For a given trajectory (1),

Given A change Δv of the velocity and a drift of the initial position $z(t_1)$.

Sought A new trajectory $\tilde{z}(t)$ in B-spline representation

$$\tilde{z}(t) = \sum_{i=1}^{n} \begin{pmatrix} c_i + \Delta c_i \\ d_i + \Delta d_i \end{pmatrix} B_i(t), \quad t \in [t_1, t_m],$$

such that $\tilde{z}'(t) = v(t) + \Delta v(t)$ and $\tilde{z}(t_1) = z(t_1) + \Delta z(t_1)$.

The procedure will turn out that the procedure amounts to such one that can be applied independently to the two components x, y of z. Therefore, we will describe it below for the x-component, only, in order to save typing.

Let $x(t) = \sum_{i=1}^{n} c_i B_i(t)$ and $\tilde{x} = x + \Delta x$, Then we have $x \in S_{k,\mathbf{t}}$ and $v = x' \in S'_{k,\mathbf{t}}$. In particular, it holds $v = P_{k,\mathbf{t}} v$. Since $\tilde{x}$ is supposed to live in $S_{k,\mathbf{t}}$, $\Delta x \in S_{k,\mathbf{t}}$ and $\Delta x' \in S'_{k,\mathbf{t}}$. This leads to the following weakening of the perturbation requirement,

$$\tilde{x}'(t) = x'(t) + \Delta x'(t) = P_{k,\mathbf{t}}(v(t) + \Delta v(t)) = v(t) + P_{k,\mathbf{t}} \Delta v(t).$$

So the first step in this procedure must be the computation of $P_{k,\mathbf{t}} \Delta v$. This will be discussed later.

Since $P_{k,\mathbf{t}} \Delta v(t) \in S'_{k,\mathbf{t}}$, there exist coefficients $\widetilde{\Delta c}_i$ such that

$$\Delta v(t) = \sum_{i=1}^{n} \widetilde{\Delta c}_i B'_i(t).$$

This leads to

$$\begin{aligned} \Delta x(t) &= \int_{t_1}^{t} \sum_{i=1}^{n} \widetilde{\Delta c}_i B'_i(\tau) d\tau + \Delta x(t_1) \\ &= \sum_{i=1}^{n} \widetilde{\Delta c}_i B_i(t) + \underbrace{\Delta x(t_1) - \sum_{i=1}^{n} \widetilde{\Delta c}_i B_i(t_1)}_{=: x_0}. \end{aligned}$$

Furthermore, it holds $\sum_{i=1}^{n} B_i(t) \equiv 1$. Hence, $x_0 = \sum_{i=1}^{n} x_0 B_i(t)$. This provides finally

$$\begin{aligned} \tilde{x}(t) &= x(t) + \Delta x(t) \\ &= \sum_{i=1}^{n} c_i B_i(t) + \sum_{i=1}^{n} \widetilde{\Delta c}_i B_i(t) + x_0 = \sum_{i=1}^{n} x_0 B_i(t) \\ &= \sum_{i=1}^{n} \left(c_i + \underbrace{\widetilde{\Delta c}_i + x_0}_{\Delta c_i} \right) B_i(t) \\ &= \sum_{i=1}^{n} (c_i + \Delta c_i) B_i(t). \end{aligned} \tag{2}$$

This solves the problem.

2 Perturbation of Velocity and Curvature

In order to understand the following considerations it is convenient to have a slightly different look on the previous derivations. The solution $\tilde{x} = x + \Delta x$ is a solution to the initial value problem $\tilde{x}' = v + \Delta v$, $\tilde{x}(t_1) = x(t_1) + \Delta x(t_1)$. The exact solution to this initial value problem is approximated by a function in the spline space $S_{k,\mathbf{t}}$ by a suitable projection. So the expression (2) is in fact the projection of the "exact" solution $\tilde{x}$.

Given A change of (the absolute value of the velocity) Δv, the curvature $\Delta\kappa$, and the initial position $\Delta z(t_1)$.

Sought A new trajectory $\tilde{z}$ in B-spline representation such that $|\tilde{z}'(t)| = |z'(t)| + \Delta v$, $\tilde{\kappa}(t) = \kappa(t) + \Delta\kappa(t)$, and $\tilde{z}(t_1) = z(t_1) + \Delta z(t_1)$.

Note: Below it will become clear that an additional initial condition will be necessary for determing $\tilde{z}$ uniquely. Moreover, for the problem to be solvable, we will need at least that $\Delta v \geq v_{\min} - |z'|$.

For a given trajectory $z = (x, y)^T$, it holds

$$|v| = \sqrt{x'^2 + y'^2}, \quad \kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

Introduce $u = \tilde{x}' = x' + \Delta x'$, $w = \tilde{y}' = y' + \Delta y'$. The conditions on $|\tilde{z}'|$ and $\tilde{\kappa}$ translate to

$$\frac{uw' - wu'}{(u^2 + w^2)^{3/2}} = r(t) \tag{3}$$

$$u^2 + w^2 = s(t) \tag{4}$$

where $r(t) = \kappa(t) + \Delta\kappa(t)$ and $s(t) = (|z'(t)| + \Delta v)^2$. The latter system (3)-(4) is a differential-algebraic equation. Its index seem to be one[1] such that an additional (scalar) initial condition is necessary for unique solvability. How to pose it appropriately?

Once (3)-(4) is solved, we have $\Delta x' = u - x'$ and $\Delta y' = w - y'$ such that the method of Section 1 can be applied.

3 Computation of the Projection $P_{k,\mathbf{t}}$

For a given Δv, we must compute a projection $P_{k,\mathbf{t}}\Delta v$ onto $S'_{k,\mathbf{t}}$. We have a set of functions $(B'_1, \ldots, B'_n)$ which span this space. However, These functions are linearly dependent! This can easily be seen by taking the derivative of the identity $\sum_{i=1}^{n} B_i(t) \equiv 1$.

The function Δv can be any (continuous) function defined on $[t_1, t_m]$. So we arrive at an approximation problem. In principle, it would be possible to

[1] This must be checked strictly! If this systen turns out to have index two, no additional initial condition is necessary for unique solvability.

use interpolation. This may, however, lead to algorithmic problems. So I would use an approximation by sampling Δv at certain points and use a least square approximation. Here, we must do it in such a way, that functions already contained in $S'_{k,\mathbf{t}}$ will not be changed. This can be (guaranteed) obtained if there are k per interval (t_j, t_{j+1}) where the t_j denoted the breakpoints of the given B-splines.[2] It is well-known that interpolation and approximation at Chebyshev nodes is close to optimal.[3] Therefore, the collocation nodes are selected as follows:

- Let k be given. Denote by $-1 < \xi_1 < \cdots < \xi_k < 1$ the Chebyshev nodes.
- On each interval (t_j, t_{j+1}), the nodes are given by shifting them

  $$t_{jl} = t_j + \frac{t_{j+1} - t_j}{2}(\xi_l + 1), \quad l = 1, \ldots, k, \quad j = 1, \ldots, m-1.$$

  All in all, there are $k(m-1)$ such points.

Note that the corresponding matrix will always be rank-deficient because of the linear dependence of the B_i's.

[2]Karolina, are you using the same notation? Or something else instead of t_j?

[3]In fact, Gauss-Legendre nodes are better suited, but slightly harder to compute.